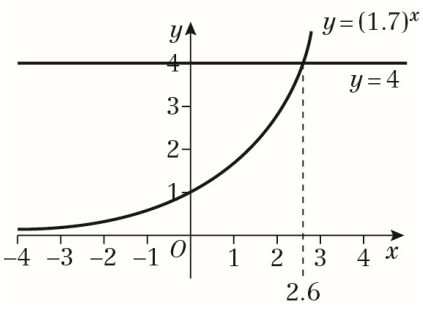
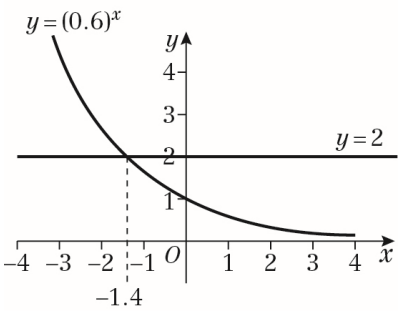
Exercise 5A

1 a



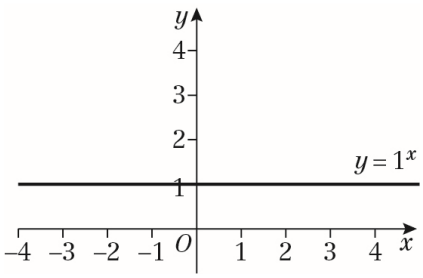
b Where $y = 4$, $x \approx 2.6$

2 a



b Where $y = 2$, $x \approx -1.4$

3

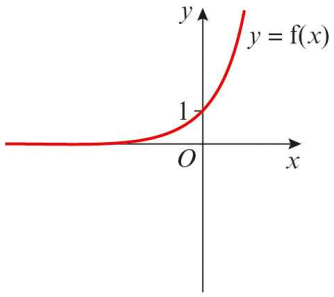


4 a True because, when $x = 0$, $a^0 = 1$ when a is positive

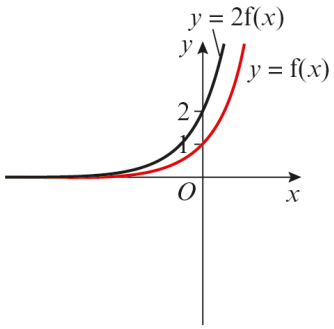
b False. For example, when $a = \frac{1}{2}$, the function $f(x) = a^x$ is not an increasing function.

c True because, when a is positive, $a^x > 0$ for all values of x.

5 a The graph crosses the y-axis when $x = 0$.
$y = 3^0$
So $y = 1$
The graph crosses the y-axis at (0, 1).
Asymptote is at $y = 0$.



b The graph is a vertical stretch by scale factor 2.
The graph crosses the y-axis when $x = 0$.
$y = 2 \times 3^0$
So $y = 2$
The graph crosses the y-axis at (0, 2).
Asymptote is at $y = 0$.



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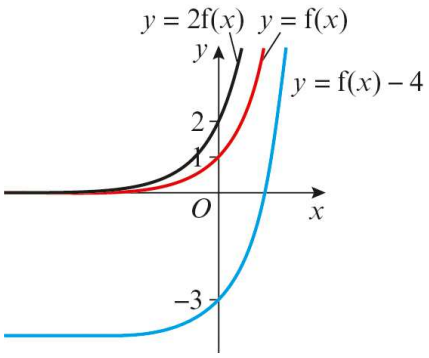
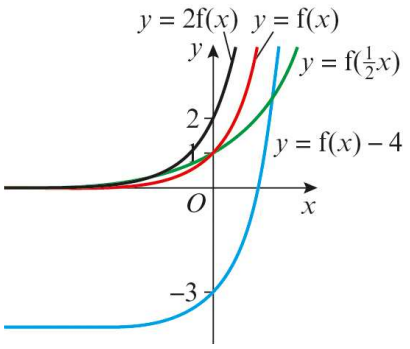
5 c The graph is a translation by the vector $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.
The graph crosses the y-axis when $x = 0$.
$y = 3^0 - 4$
So $y = -3$
The graph crosses the y-axis at $(0, -3)$.
Asymptote is at $y = -4$.



d The graph is a horizontal stretch by scale factor 2.
The graph crosses the y-axis when $x = 0$.
$y = 3^{\frac{1}{2} \times 0}$
So $y = 1$
The graph crosses the y-axis at $(0, 1)$.
Asymptote is at $y = 0$.

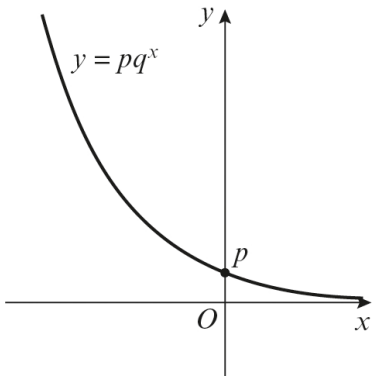


6 Substitute the coordinates into $y = ka^x$.
$6 = ka^1$ (equation 1)
$48 = ka^4$ (equation 2)
Solve simultaneously: divide equation 2 by equation 1,

$$48 \div 6 = \frac{ka^4}{ka}$$

$a^3 = 8$
$a = 2, k = 3$

7 a As x increases, y decreases



b Substitute the coordinates into $y = pq^x$.
$150 = pq^{-3}$ (equation 1)
$0.048 = pq^2$ (equation 2)
Solve simultaneously, divide equation 2 by equation 1.

$$0.048 \div 150 = \frac{pq^2}{pq^{-3}}$$

$q^5 = 0.00032$
$q = 0.2$
$p = 0.048 \div 0.2^2 = 1.2$
$p = 1.2, q = 0.2$

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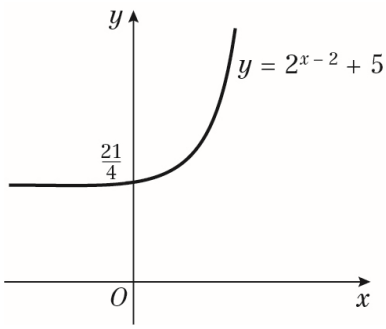
Challenge

To draw the graph, note that it is a translation of the graph $y = 2^x$ by the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

The graph crosses the y-axis when $x = 0$,
so $y = 2^{0-2} + 5$
$y = 5.25$
The graph crosses the y-axis at (0, 5.25).
Asymptote is at $y = 5$.



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Exercise 5B

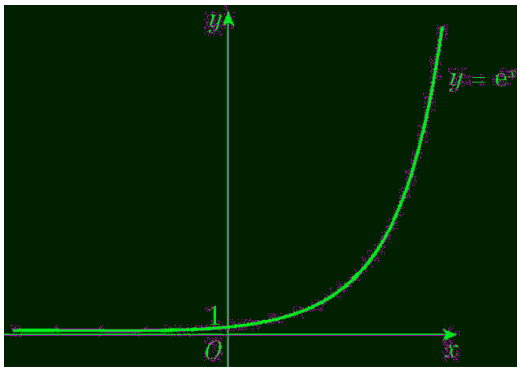
1 a 2.71828

b 54.59815

c 0.00005

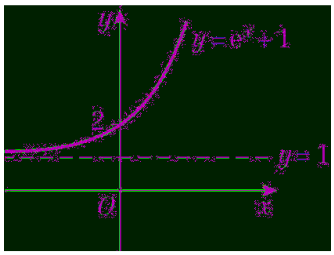
d 1.22140

2 a



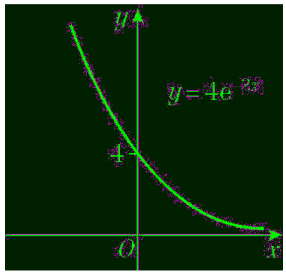
c $e = 2.71828\ldots$
$e^3 = 20.08553\ldots$

3 a $y = e^x + 1$



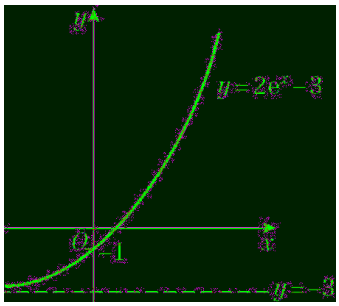
This is the usual $y = e^x$ 'moved up' (translated) 1 unit

3 b $y = 4e^{-2x}$



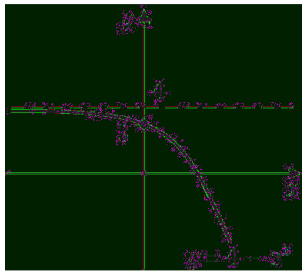
$x = 0 \Rightarrow y = 4$
As $x \to -\infty, y \to \infty$
As $x \to \infty, y \to 0$
This is an exponential decay type of graph.

c $y = 2e^x - 3$



$x = 0 \Rightarrow y = 2 \times 1 - 3 = -1$
As $x \to \infty,\ y \to \infty$
As $x \to -\infty,\ y \to 2 \times 0 - 3 = -3$

d $y = 4 - e^x$



$x = 0 \Rightarrow y = 4 - 1 = 3$
As $x \to \infty,\ y \to 4 - \infty$, i.e. $y \to -\infty$
As $x \to -\infty,\ y \to 4 - 0 = 4$

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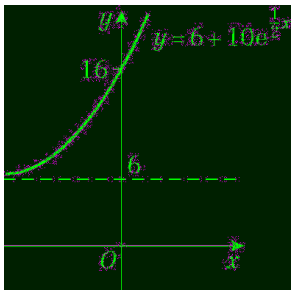
3 e $y = 6 + 10e^{\frac{1}{2}x}$

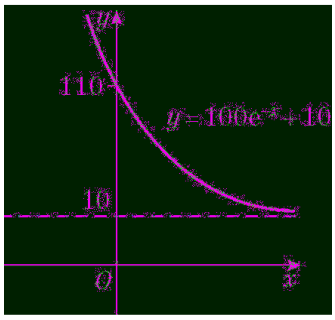


$x = 0 \Rightarrow y = 6 + 10 \times 1 = 16$

As $x \to \infty, y \to \infty$

As $x \to -\infty, y \to 6 + 10 \times 0 = 6$

f $y = 100e^{-x} + 10$



$x = 0 \Rightarrow y = 100 \times 1 + 10 = 110$

As $x \to \infty,\ y \to 100 \times 0 + 10 = 10$

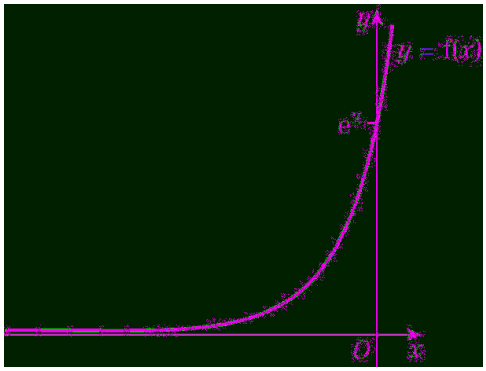
As $x \to -\infty,\ y \to \infty$

4 a The graph is increasing so b is positive.
The line $y = 5$ is an asymptote, so $C = 5$.
When $x = 0$, $6 = Ae^{b \times 0} + C = A + 5$, so $A = 1$.

b The graph is decreasing so b is negative.
The line $y = 0$ is an asymptote, so $C = 0$.
When $x = 0$, $4 = Ae^{b \times 0} + C = A + 0$, so $A = 4$.

4 c The graph is increasing so b is positive.
The line $y = 2$ is an asymptote, so $C = 2$.
When $x = 0$, $8 = Ae^{b \times 0} + C = A + 2$, so $A = 6$.

5 $f(x) = e^{3x+2}$
$= e^{3x} \times e^2$
$= e^2e^{3x}$
$A = e^2$ and $b = 3$



6 a $y = e^{6x}$

$\frac{dy}{dx} = 6e^{6x}$

b $y = e^{-\frac{1}{3}x}$

$\frac{dy}{dx} = -\frac{1}{3}e^{-\frac{1}{3}}$

c $y = 7e^{2x}$

$\frac{dy}{dx} = 2 \times 7e^{2x} = 14e^{2x}$

d $y = 5e^{0.4x}$

$\frac{dy}{dx} = 0.4 \times 5e^{0.4x} = 2e^{0.4x}$

e $y = e^{3x} + 2e^x$

$\frac{dy}{dx} = 3e^{3x} + 2e^x$

f $y = e^x(e^x + 1) = e^{2x} + e^x$

$\frac{dy}{dx} = 2e^{2x} + e^x$

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7 a $y = e^{3x}$

$\frac{dy}{dx} = 3e^{3x}$

When $x = 2$,

$\frac{dy}{dx} = 3e^{3 \times 2} = 3e^{6}$

b When $x = 0$,

$\frac{dy}{dx} = 3e^{3 \times 0} = 3$

c When $x = -0.5$,

$\frac{dy}{dx} = 3e^{3 \times -0.5} = 3e^{-1.5}$

8 $f(x) = e^{0.2x}$

$f'(x) = 0.2e^{0.2x}$

The gradient of the tangent when $x = 5$

is $f'(5) = 0.2e^{0.2 \times 5} = 0.2e$

$f(5) = e^{0.2 \times 5} = e$

The equation of the tangent in the form

$y = mx + c$

is $e = 0.2e \times 5 + c$

$e = e + c$

so $c = 0$

Therefore the tangent to the curve at the point $(5, c)$ is in the form $y = mx$.

Thus it so goes through the origin.

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Exercise 5C

1 a When $e^x = 6$
$\ln(e^x) = \ln 6$
$x = \ln 6$

b When $e^{2x} = 11$
$\ln(e^{2x}) = \ln 11$
$2x = \ln 11$
$x = \frac{1}{2}\ln 11$

c When $e^{-x+3} = 20$
$\ln(e^{-x+3}) = \ln 20$
$-x + 3 = \ln 20$
$x = 3 - \ln 20$

d When $3e^{4x} = 1$
$e^{4x} = \frac{1}{3}$
$\ln(e^{4x}) = \ln \frac{1}{3}$
$4x = \ln \frac{1}{3}$
$x = \frac{1}{4}\ln \frac{1}{3}$

e When $e^{2x+6} = 3$
$\ln(e^{2x+6}) = \ln 3$
$2x + 6 = \ln 3$
$x = \ln 3 - 6$
$x = \frac{1}{2}\ln 3 - 3$

f When $e^{5-x} = 19$
$\ln(e^{5-x}) = \ln 19$
$5 - x = \ln 19$
$x = 5 - \ln 19$

2 a When $\ln x = 2$
$e^{\ln x} = e^2$
$x = e^2$

b When $\ln(4x) = 1$
$e^{\ln(4x)} = e^1$
$4x = e^1$
$x = \frac{e}{4}$

c When $\ln(2x + 3) = 4$
$e^{\ln(2x+3)} = e^4$
$2x + 3 = e^4$
$2x = e^4 - 3$
$x = \frac{1}{2}e^4 - \frac{3}{2}$

2 d When $2\ln(6x - 2) = 5$
$\ln(6x - 2) = \frac{5}{2}$
$e^{\ln(6x-2)} = e^{\frac{5}{2}}$
$6x - 2 = e^{\frac{5}{2}}$
$6x = e^{\frac{5}{2}} + 2$
$x = \frac{1}{6}(e^{\frac{5}{2}} + 2)$

e When $\ln(18 - x) = \frac{1}{2}$
$e^{\ln(18-x)} = e^{\frac{1}{2}}$
$18 - x = e^{\frac{1}{2}}$
$x = 18 - e^{\frac{1}{2}}$

f When $\ln(x^2 - 7x + 11) = 0$
$e^{\ln(x^2-7x+11)} = e^0$
$x^2 - 7x + 11 = 1$
$x^2 - 7x + 10 = 0$
$(x - 2)(x - 5) = 0$
$x = 2$ or $x = 5$

3 a $e^{2x} - 8e^x + 12 = 0$
Let $u = e^x$
$u^2 - 8u + 12 = 0$
$(u - 2)(u - 6) = 0$
$u = 2$ or $u = 6$
$e^x = 2$ or $e^x = 6$

When $e^x = 2$
$\ln(e^x) = \ln 2$
$x = \ln 2$
When $e^x = 6$
$\ln(e^x) = \ln 6$
$x = \ln 6$

$x = \ln 2$ or $x = \ln 6$

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3 b $e^{4x} - 3e^{2x} + 2 = 0$

Let $u = e^{2x}$

$u^2 - 3u + 2 = 0$

$(u - 1)(u - 2) = 0$

$u = 1$ or $u = 2$

$e^{2x} = 1$ or $e^{2x} = 2$

When $e^{2x} = 1$

$\ln(e^{2x}) = \ln 1$

$2x = 0$

$x = 0$

When $e^{2x} = 2$

$\ln(e^{2x}) = \ln 2$

$2x = \ln 2$

$x = \frac{1}{2} \ln 2$

$x = 0$ or $x = \frac{1}{2} \ln 2$

c $(\ln x)^2 + 2\ln x - 15 = 0$

Let $u = \ln x$

$u^2 + 2u - 15 = 0$

$(u + 5)(u - 3) = 0$

$u = -5$ or $u = 3$

When $\ln x = -5$

$e^{\ln x} = e^{-5}$

$x = e^{-5}$

When $\ln x = 3$

$e^{\ln x} = e^{3}$

$x = e^{3}$

$x = e^{-5}$ or $x = e^{3}$

d $e^{x} - 5 + 4e^{-x} = 0$

Multiply each term by e^x

$e^{2x} - 5e^{x} + 4 = 0$

Let $u = e^x$

$u^2 - 5u + 4 = 0$

$(u - 1)(u - 4) = 0$

$u = 1$ or $u = 4$

$e^{x} = 1$ or $e^{x} = 4$

When $e^{x} = 1$

$\ln(e^{x}) = \ln 1$

$x = 0$

When $e^{x} = 4$

$\ln(e^{x}) = \ln 4$

$x = \ln 4$

$x = 0$ or $x = \ln 4$

3 e $3e^{2x} - 16e^{x} + 5 = 0$

Let $u = e^x$

$3u^2 - 16u + 5 = 0$

$(3u - 1)(u - 5) = 0$

$u = \frac{1}{3}$ or $u = 5$

$e^{x} = \frac{1}{3}$ or $e^{x} = 5$

When $e^{x} = \frac{1}{3}$

$\ln(e^{x}) = \ln \frac{1}{3}$

$x = \ln \frac{1}{3}$

When $e^{x} = 5$

$\ln(e^{x}) = \ln 5$

$x = \ln 5$

$x = \ln \frac{1}{3}$ or $x = \ln 5$

f $(\ln x)^2 - 4\ln x - 12 = 0$

Let $u = \ln x$

$u^2 - 4u - 12 = 0$

$(u + 2)(u - 6) = 0$

$u = -2$ or $u = 6$

When $\ln x = -2$

$e^{\ln x} = e^{-2}$

$x = e^{-2}$

When $\ln x = 6$

$e^{\ln x} = e^{6}$

$x = e^{6}$

$x = e^{-2}$ or $x = e^{6}$

4 $e^{x} - 7 + 12e^{-x} = 0$

Multiply each term by e^x

$e^{2x} - 7e^{x} + 12 = 0$

Let $u = e^x$

$u^2 - 7u + 12 = 0$

$(u - 3)(u - 4) = 0$

$u = 3$ or $u = 4$

$e^{x} = 3$ or $e^{x} = 4$

When $e^{x} = 3$

$\ln(e^{x}) = \ln 3$

$x = \ln 3$

When $e^{x} = 4$

$\ln(e^{x}) = \ln 4$

$x = \ln 2^2$

$x = 2 \ln 2$

$x = \ln 3$ or $x = 2 \ln 2$

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5 a When $\ln(8x-3)=2$

$$e^{\ln(8x-3)}=e^2$$
$$8x-3=e^2$$
$$8x=e^2+3$$
$$x=\tfrac{1}{8}(e^2+3)$$

b When $e^{5(x-8)}=3$

$$\ln(e^{5(x-8)})=\ln 3$$
$$5(x-8)=\ln 3$$
$$x-8=\tfrac{1}{5}\ln 3$$
$$x=\tfrac{1}{5}\ln 3+8$$

c $e^{10x}-8e^{5x}+7=0$

Let $u=e^{5x}$

$$u^2-8u+7=0$$
$$(u-1)(u-7)=0$$
$$u=1 \text{ or } u=7$$
$$e^{5x}=1 \text{ or } e^{5x}=7$$

When $e^{5x}=1$

$$\ln(e^{5x})=\ln 1$$
$$5x=0$$
$$x=0$$

When $e^{5x}=7$

$$\ln(e^{5x})=\ln 7$$
$$5x=\ln 7$$
$$x=\tfrac{1}{5}\ln 7$$

$x=0$ or $x=\tfrac{1}{5}\ln 7$

d When $(\ln x-1)^2=4$

$$(\ln x)^2-2\ln x-3=0$$

Let $u=\ln x$

$$u^2-2u-3=0$$
$$(u+1)(u-3)=0$$
$$u=-1 \text{ or } u=3$$

When $\ln x=-1$

$$e^{\ln x}=e^{-1}$$
$$x=e^{-1}$$

When $\ln x=3$

$$e^{\ln x}=e^{3}$$
$$x=e^{3}$$

$x=e^{-1}$ or $x=e^3$

6 When $3^x e^{4x-1}=5$

$$\ln(3^x e^{4x-1})=\ln 5$$
$$\ln(3^x)+\ln(e^{4x-1})=\ln 5$$
$$x\ln 3+4x-1=\ln 5$$
$$x\ln 3+4x=1+\ln 5$$
$$x(\ln 3+4)=1+\ln 5$$
$$x=\frac{1+\ln 5}{4+\ln 3}$$

7 a $D=6$ when $t=0$ so 6 is the initial concentration of the drug in mg/l.

b $D=6e^{\frac{-t}{10}}$

When $t=2$

$$D=6e^{\frac{-2}{10}}$$
$$D=4.91 \text{ mg/l (3 s.f.)}$$

c When $6e^{\frac{-t}{10}}=3$

$$e^{\frac{-t}{10}}=\tfrac{1}{2}$$
$$\ln e^{\frac{-t}{10}}=\ln\tfrac{1}{2}$$
$$-\tfrac{1}{10}t=\ln\tfrac{1}{2}$$
$$t=-10\ln\tfrac{1}{2}$$
$$t=6.931471...$$

$t=6$ hours and 55.888... minutes

$t=6$ hours and 56 minutes

8 a A is where $x=0$

Substitute $x=0$ into $y=3+\ln(4-x)$ to give

$$y=3+\ln 4$$
$$A=(0,3+\ln 4)$$

b B is where $y=0$

Substitute $y=0$ into $y=3+\ln(4-x)$ to give

$$0=3+\ln(4-x)$$
$$-3=\ln(4-x)$$
$$e^{-3}=4-x$$
$$x=4-e^{-3}$$
$$B=(4-e^{-3},0)$$

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Challenge

$g(0) = Ae^{B \times 0} + C = 5$
$A + C = 5$
As $y = 2$ is an asymptote, $C = 2$
$A = 3$ and $g(6) = 3e^{B \times 6} + 2 = 10$
$3e^{6B} = 8$
$e^{6B} = \frac{8}{3}$
$\ln(e^{6B}) = \ln\frac{8}{3}$
$6B = \ln\frac{8}{3}$
$B = \frac{1}{6}\ln\frac{8}{3}$

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Exercise 5D

1 a When $S = 4 \times 7^x$
$\log S = \log(4 \times 7^x)$
$\log S = \log 4 + \log 7^x$
$\log S = \log 4 + x \log 7$

b $\log S = x \log 7 + \log 4$
Gradient $= \log 7$
Intercept $= \log 4$

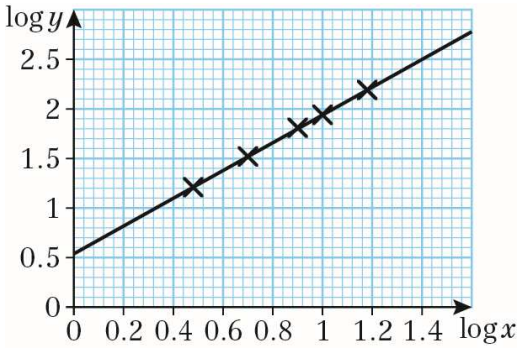
2 a When $A = 6x^4$
$\log A = \log(6x^4)$
$\log A = \log 6 + \log x^4$
$\log A = \log 6 + 4 \log x$

b $\log A = 4 \log x + \log 6$
Gradient $= 4$
Intercept $= \log 6$

3 a

$\log x$	0.48	0.70	0.90	1	1.18
$\log y$	1.21	1.52	1.81	1.94	2.19

b



c $y = ax^n$
$\log y = \log(ax^n)$
$\log y = \log a + \log x^n$
$\log y = \log a + n \log x$
$\log y = n \log x + \log a$
Gradient $= n$
Intercept $= \log a$
Calculating the gradient from the table,

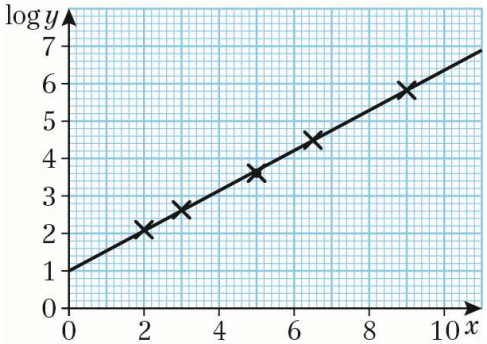
$$n = \frac{2.19 - 1.21}{1.18 - 0.48} = \frac{0.98}{0.7} = 1.4$$

Reading the intercept from the graph,
$\log a = 0.55$
$a = 10^{0.55} = 3.548...$
$a = 3.5,\ n = 1.4$

4 a

x	2	3	5	6.5	9
$\log y$	2.10	2.63	3.61	4.49	5.82

b



c $y = ab^x$
$\log y = \log(ab^x)$
$\log y = \log a + \log b^x$
$\log y = \log a + x \log b$
$\log y = x \log b + \log a$
Gradient $= \log b$
Intercept $= \log a$
Calculating the gradient from the table and the graph,

$$\log b = \frac{5.82 - 1}{9 - 0} = \frac{4.82}{9} = 0.53555...$$

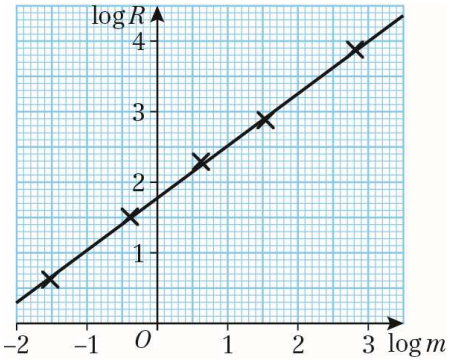
$$b = 10^{0.53555...} = 3.43...$$

Reading the intercept from the graph,
$\log a = 1$
$a = 10^1 = 10$
$a = 10,\ b = 3.4$

5 a

$\log m$	−1.52	−0.39	0.62	1.54	2.81
$\log R$	0.62	1.51	2.29	2.88	3.88

b



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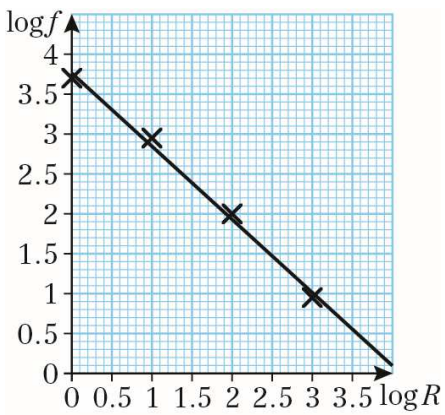
5 c $R = am^b$
$\log R = \log(am^b)$
$\log R = \log a + \log m^b$
$\log R = \log a + b \log m$
Gradient $= b$
Intercept $= \log a$
Calculating the gradient from the table,
$$b = \frac{3.88 - 0.62}{2.81 - (-1.52)} = \frac{3.26}{4.33} = 0.75288...$$
Reading the intercept from the graph,
$\log a = 1.78$
$a = 10^{1.78} = 60.255...$
$a = 60,\ b = 0.75$

d $R = 60m^{0.75}$
When $m = 80$
$R = 60(80)^{0.75} = 1604.97...$
1605 kcal/day

6 a

$\log R$	0	1	2	3
$\log f$	3.69	2.94	1.96	0.95

b



c $f = AR^b$
$\log f = \log(AR^b)$
$\log f = \log A + \log R^b$
$\log f = \log A + b \log R$
$\log y = b \log R + \log A$
Gradient $= b$
Intercept $= \log A$
Calculating the gradient from the table,
$$b = \frac{0.95 - 3.69}{3 - 0} = \frac{-2.74}{3} = -0.91...$$
Reading the intercept from the graph,
$\log A = 3.76$
$A = 10^{3.76} = 5754.39...$
$A = 5800,\ b = -0.9$

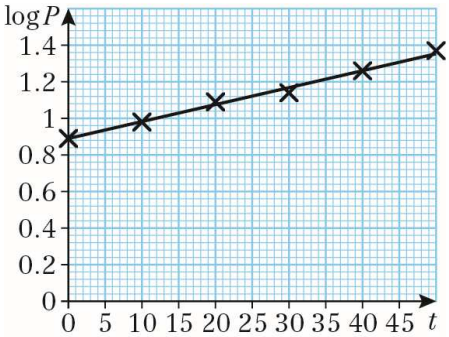
6 d $f = 5800R^{-0.9}$ per 100 000 words
When $R = 57$
$f = 152.45...$
For 455 125 words, $4.55125 \times f = 693.85...$
694 times

7 a

t	0	10	20	30	40	50
$\log P$	0.88	0.98	1.08	1.13	1.26	1.37

b When $P = ab^t$
$\log P = \log(ab^t)$
$\log P = \log a + \log b^t$
$\log P = \log a + t \log b$

c



d Gradient $= \log b$
Intercept $= \log a$
Calculating the gradient from the table,
$$\log b = \frac{1.37 - 0.88}{50 - 0} = \frac{0.49}{50} = 0.0098$$
$$b = 10^{0.0098} = 1.0228...$$
Reading the intercept from the graph,
$\log a = 0.88$
$a = 10^{0.88} = 7.5857...$
$a = 7.59,\ b = 1.03$

e The rate of growth is often proportional to the size of the population

8 a $N = ab^t$
$\log N = \log(ab^t)$
$\log N = \log a + \log b^t$
$\log N = \log a + t \log b$
$$\text{Gradient} = \frac{2.55 - 1.6}{10 - 0} = \frac{0.95}{10} = 0.095$$
Intercept $= 1.6$
$\log N = 0.095t + 1.6$

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8 b $\log a = 1.6$
$a = 10^{1.6} = 39.8...$
$\log b = 0.095$
$b = 10^{0.095} = 1.2445...$
$a = 40, b = 1.2$

c a is the initial number of sick people

d $N = ab^t$
$N = 40(1.2)^{30} = 9495.052 = 9500$ (2 s.f.)
After 30 days people may start to recover, or the disease may stop spreading as quickly.

9 a $A = pw^q$
$\log A = m \log w + c$
Intercept $= -0.1049$
Gradient $= 2$
$\log A = 2 \log w - 0.1049$

b $A = pw^q$
$\log A = \log (pw^q)$
$\log A = \log p + \log w^q$
$\log A = \log p + q \log w$
Equating coefficients
$q = 2$
$\log p = -0.1049$
$p = 10^{-0.1049}$
$p = 0.785416...$

c The shapes are circles.
Multiply p by 4
$4p = 3.1416... \approx \pi$
So p is approximately $\frac{1}{4}$ of π

So $A = \frac{\pi}{4} w^2$
The width is the diameter of the circle
so $A = \frac{\pi}{4}(2r)^2 = \pi r^2$

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Exercise 5E

1 $V = 20\ 000\,e^{-\frac{1}{12}}$

a The new value is when $t = 0$

$\Rightarrow V = 20\ 000 \times e^{-\frac{0}{12}}$

$= 20\ 000 \times 1$

$= 20\ 000$

New value = €20 000

b Value after 4 years is given when $t = 4$

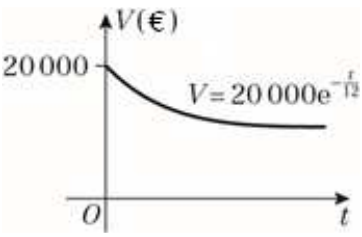
$\Rightarrow V = 20\ 000 \times e^{-\frac{4}{12}}$

$= 20\ 000 \times e^{-\frac{1}{3}}$

$= 14\ 330.63$

Value after 4 years is €14 331 (to nearest €).

c



2 $P = 20 + 10\,e^{\frac{t}{50}}$

a The year 2000 corresponds to $t = 0$.

Substitute $t = 0$ into $P = 20 + 10\,e^{\frac{t}{50}}$

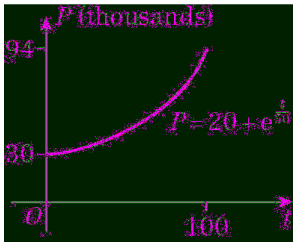
$P = 20 + 10 \times e^{0} = 20 + 10 \times 1 = 30$

Population = 30 thousand

b $P = 20 + 10e^{\frac{30}{50}}$

$P = 38.221...$ thousand

2 c



Year 2100 is $t = 100$

$P = 20 + 10\,e^{\frac{100}{50}}$

$= 20 + 10\,e^{2}$

$= 93.891$ thousand

d $P = 20 + 10e^{\frac{500}{50}}$

$= 220\ 284.6579...$ thousand

The model predicts the population of the country to be over 220 million. This is highly unlikely and by 2500 new factors are likely to affect population growth. Therefore, the model is not valid for predictions that far into the future.

3 $N = 300 - 100\,e^{-0.5t}$

a The number first diagnosed means when $t = 0$.

Substitute $t = 0$ in $N = 300 - 100\,e^{-0.5t}$

$N = 300 - 100 \times e^{-0.5 \times 0}$

$= 300 - 100 \times 1$

$= 200$

b The long term prediction suggests $t \to \infty$.

As $t \to \infty, e^{-0.5t} \to 0$

So $N \to 300 - 100 \times 0 = 300$

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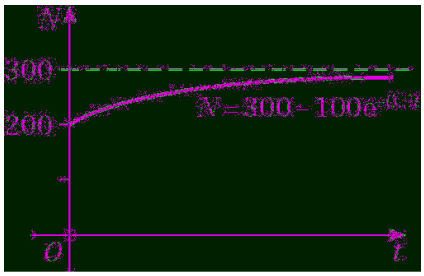
3 c



4 a i $R = 12e^{0.2m}$
$R = 12e^{0.2 \times 1}$
$= 14.6568...$
15 rabbits

ii $R = 12e^{0.2m}$
$R = 12e^{0.2 \times 12}$
$= 132.278...$
132 rabbits

b When $m = 0$, $R = 12$
12 is the initial number of rabbits in the population.

c $\frac{dR}{dm} = 0.2 \times 12e^{0.2m}$
$= 2.4e^{0.2m}$
When $m = 6$,
$\frac{dR}{dm} = 2.4e^{0.2 \times 6}$
$= 7.9682...$
≈ 8

d This model will stop giving valid results for large enough values of t as new factors are likely to affect population growth, such as the rabbits running out of food or space.

5 a $p = e^{-0.13h}$
$p = e^{-0.13 \times 4.394}$
$= 0.5648359...$
0.565 bars

5 b $\frac{dp}{dh} = -0.13e^{-0.13h}$
As $p = e^{-0.13h}$
$\frac{dp}{dh} = -0.13p$
$k = -0.13$

c The atmospheric pressure decreases exponentially as the altitude increases.

d $\frac{dp}{dh} = -0.13e^{-0.13h}$
When $h = 0$,
$\frac{dp}{dh} = -0.13e^{-0.13 \times 0}$
$= -0.13$
When $h = 1$,
$\frac{dp}{dh} = -0.13e^{-0.13 \times 1}$
$= -0.114152406...$

Percentage change
$= \frac{-0.11415206 + 0.13}{-0.13} \times 100$
$= 12.19...$
$= 12\%$

6 a Model 1: $V = 20\,000e^{-0.24t}$
When $t = 1$, $V = 20\,000e^{-0.24 \times 1}$
$= 15\,732.557...$
$= 15\,733$ Dirhams

Model 2: $V = 19\,000e^{-0.255t} + 1000$
When $t = 1$, $V = 19\,000e^{-0.255 \times 1} + 1000$
$= 15\,723.413...$
$= 15\,723$ Dirhams

So, Model 1 predicts a larger value.

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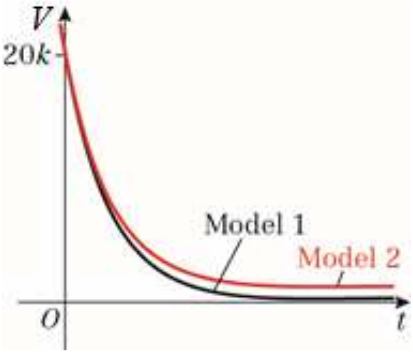
6 b Model 1: $V = 20\,000e^{-0.24t}$
When $t = 10$, $V = 20\,000e^{-0.24 \times 10}$
$= 1814.359...$
$= 1814$ Dirhams

Model 2: $V = 19\,000e^{-0.255t} + 1000$
When $t = 10$,
$V = 19\,000e^{-0.255 \times 10} + 1000$
$= 2483.5516...$
$= 2484$ Dirhams

So, Model 2 predicts a larger value.

c



d In Model 2 the car will always be worth at least 1000 Dirhams. This could be the value of the car as scrap metal.

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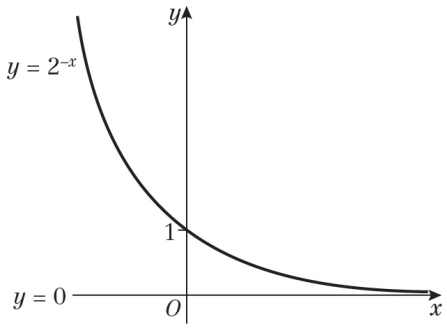
Chapter review 5

1 a $y = 2^{-x} = (2^{-1})^x = (\frac{1}{2})^x$



b $y = 5e^x - 1$

The graph is a translation by the vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and a vertical stretch scale factor 5 of the graph $y = e^x$.

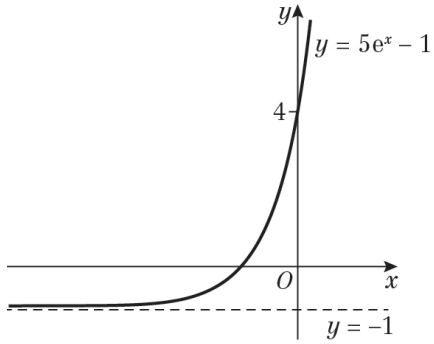
The graph crosses the y-axis when $x = 0$.

$y = 5 \times e^0 - 1$

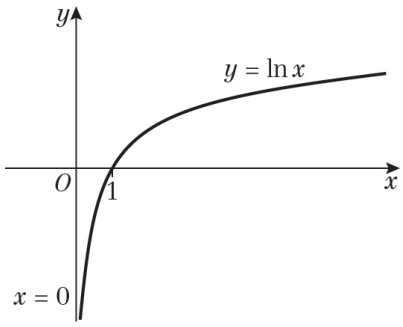
$y = 4$

The graph crosses the y-axis at (0, 4).

Asymptote is at $y = -1$.



c $y = \ln x$



2 a $\ln(p^2q) = \ln(p^2) + \ln(q)$

$= 2\ln(p) + \ln(q)$

b $\ln(pq) = 5$ and $\ln(p^2q) = 9$

$\ln(pq) = \ln(p) + \ln(q) = 5$ **(1)**

$\ln(p^2q) = 2\ln(p) + \ln(q) = 9$ **(2)**

Subtracting equation **(1)** from equation **(2)** gives:

$\ln(p) = 4$

Substituting into equation **(1)** gives:

$4 + \ln(q) = 5$

Therefore

$\ln(q) = 1$

3 a $y = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$

b $y = e^{11x}$

$\frac{dy}{dx} = 11e^{11x}$

c $y = 6e^{5x}$

$\frac{dy}{dx} = 5 \times 6e^{5x} = 30e^{5x}$

4 a $\ln(2x - 5) = 8$ (inverse of ln)

$2x - 5 = e^8$ $(+5)$

$2x = e^8 + 5$ $(\div 2)$

$x = \frac{e^8 + 5}{2}$

b $e^{4x} = 5$ (inverse of e)

$4x = \ln 5$ $(\div 4)$

$x = \frac{\ln 5}{4}$

c $24 - e^{-2x} = 10$ $(+e^{-2x})$

$24 = 10 + e^{-2x}$ (-10)

$14 = e^{-2x}$ (inverse of e)

$\ln(14) = -2x$ $(\div -2)$

$-\frac{1}{2}\ln(14) = x$

$x = -\frac{1}{2}\ln(14)$

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4 d $\ln(x)+\ln(x-3)=0$

$\ln(x(x-3))=0$

$x(x-3)=e^0$

$x(x-3)=1$

$x^2-3x-1=0$

$x=\dfrac{3\pm\sqrt{9+4}}{2}$

$=\dfrac{3\pm\sqrt{13}}{2}$

$=\dfrac{3+\sqrt{13}}{2}$

(x cannot be negative because of initial equation)

e $e^x+e^{-x}=2$

$e^x+\dfrac{1}{e^x}=2 \quad (\times\ e^x)$

$(e^x)^2+1=2e^x$

$(e^x)^2-2e^x+1=0$

$(e^x-1)^2=0$

$e^x=1$

$x=\ln 1=0$

f $\ln 2+\ln x=4$

$\ln 2x=4$

$2x=e^4$

$x=\dfrac{e^4}{2}$

5 $P=100+850e^{-\frac{t}{2}}$

a New price is when $t=0$

Substitute $t=0$ into $P=100+850e^{-\frac{t}{2}}$ to give:

$P=100+850e^{-\frac{0}{2}} \quad (e^0=1)$

$=100+850=950$

The new price is €950

5 b After 3 years $t=3$.

Substitute $t=3$ into $P=100+850e^{-\frac{t}{2}}$ to give:

$P=100+850e^{-\frac{3}{2}}=289.66$

Price after 3 years is €290 (to nearest €)

c It is worth less than €200 when $P<200$

Substitute $P=200$ into $P=100+850e^{-\frac{t}{2}}$ to give:

$200=100+850e^{-\frac{t}{2}}$

$100=850e^{-\frac{t}{2}}$

$\dfrac{100}{850}=e^{-\frac{t}{2}}$

$\ln\left(\dfrac{100}{850}\right)=-\dfrac{t}{2}$

$t=-2\ln\left(\dfrac{100}{850}\right)$

$t=4.28$

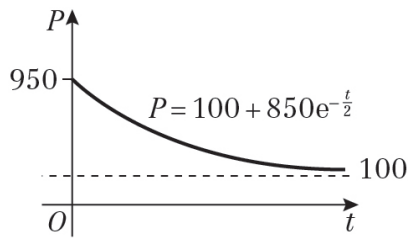
It is worth less than €200 after 4.28 years.

d As $t\to\infty$, $e^{-\frac{t}{2}}\to 0$

Hence, $P\to 100+850\times 0=100$

The computer will be worth €100 eventually.

e



f A good model. The computer will always be worth something.

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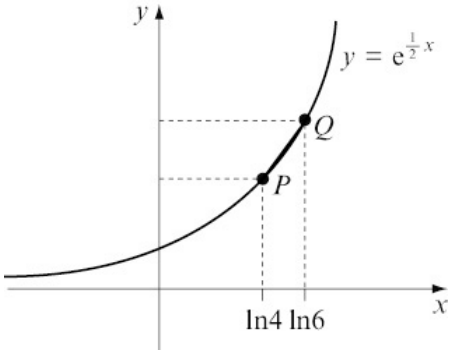
6 a



Q has y-coordinate $e^{\frac{1}{2}\ln 16} = e^{\ln 16 \times \frac{1}{2}} = 16^{\frac{1}{2}} = 4$

P has y-coordinate $e^{\frac{1}{2}\ln 4} = e^{\ln 4 \times \frac{1}{2}} = 4^{\frac{1}{2}} = 2$

$$\text{Gradient of the line } PQ = \frac{\text{change in } y}{\text{change in } x} = \frac{4-2}{\ln 16 - \ln 4} = \frac{2}{\ln\left(\frac{16}{4}\right)} = \frac{2}{\ln 4}$$

Using $y = mx + c$, the equation of the line PQ is:

$$y = \frac{2}{\ln 4}x + c$$

$(\ln 4, 2)$ lies on the line so

$$y = \frac{2}{\ln 4}x + c$$

$$2 = 2 + c$$

$$c = 0$$

Equation of PQ is $y = \dfrac{2x}{\ln 4}$

b The line passes through the origin as $c = 0$.

c Length from $(\ln 4, 2)$ to $(\ln 16, 4)$ is

$$\sqrt{(\ln 16 - \ln 4)^2 + (4-2)^2}$$

$$= \sqrt{\left(\ln \tfrac{16}{4}\right)^2 + 2^2}$$

$$= \sqrt{(\ln 4)^2 + 4} = 2.43$$

7 a $T = 55e^{-\frac{t}{8}} + 20$

t is the time in minutes and time cannot be negative as you can't go back in time.

b The starting temperature of the cup of tea is when $t = 0$

$T = 55e^{-\frac{0}{8}} + 20 = 75°\text{C}$

c When $T = 50°\text{C}$

$$55e^{-\frac{t}{8}} + 20 = 50$$

$$55e^{-\frac{t}{8}} = 30$$

$$e^{-\frac{t}{8}} = \frac{30}{55}$$

$$\ln\left(e^{-\frac{t}{8}}\right) = \ln\left(\frac{30}{55}\right)$$

$$-\frac{t}{8} = \ln\left(\frac{30}{55}\right)$$

$$t = -8\ln\left(\frac{30}{55}\right)$$

$$= 4.849...$$

$$\approx 5 \text{ minutes}$$

d The exponential term will always be positive, so the overall temperature will be greater than 20°C.

8 a As $S = aV^b$

$\log S = \log(aV^b)$

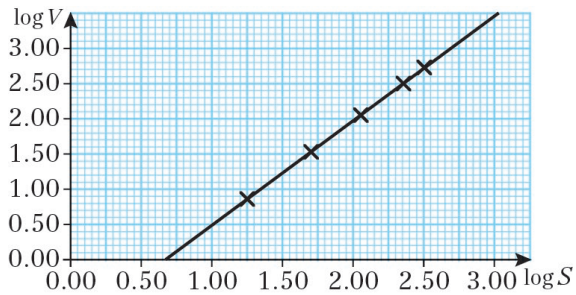
$\log S = \log a + \log(V^b)$

$\log S = \log a + b \log V$

b

$\log S$	1.26	1.70	2.05	2.35	2.50
$\log V$	0.86	1.53	2.05	2.49	2.72

c



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8 d b is the gradient $= \dfrac{2.72-0.86}{2.5-1.26}$

$= \dfrac{1.86}{1.24} = 1.5$

Intercept $= \log a$

$\log a = -1.05$

$10^{-1.05} = a$

$a = 0.0891...$

$a \approx 0.09$

9 a The student goes wrong in line 2, where the subtraction should be a division (as in line 2 below).

b The full working should have looked like this:

$$\log_2 x - \tfrac{1}{2}\log_2(x+1) = 1$$

$$\log_2 x - \log_2\left((x+1)^{\frac{1}{2}}\right) = 1$$

$$\log_2 x - \log_2(\sqrt{x+1}) = 1$$

$$\log_2 \frac{x}{\sqrt{x+1}} = 1$$

$$\frac{x}{\sqrt{x+1}} = 2^1$$

$$x = 2\sqrt{x+1} \quad \text{(square)}$$

$$x^2 = 4x+4$$

$$x^2 - 4x - 4 = 0 \quad \text{(use quadratic formula)}$$

$$x = 2 + 2\sqrt{2}$$

($x \neq 2 - 2\sqrt{2}$ because log cannot take negative input values)

10 a The gradient is given by:

$$m = \frac{\log_{10} P_2 - \log_{10} P_1}{t_2 - t_1}$$

$$= \frac{2.2-2}{20-0}$$

$$= 0.01$$

The line crosses the vertical axis at (0, 2) therefore the equation of the line is:

$\log_{10} P = 0.01t + 2$

10 b $P = ab^t$

$$\log_{10}(P) = \log_{10}(ab^t)$$

$$= \log_{10}(a) + \log_{10}(b^t)$$

$$= \log_{10}(a) + t\log_{10}(b)$$

Comparing

$\log_{10}(P) = \log_{10}(a) + t\log_{10}(b)$

to

$\log_{10} P = 0.01t + 2$

gives:

$\log_{10}(a) = 2 \Rightarrow a = 100$

When $t = 0$, $P_0 = ab^0 = a$

Therefore, a represents the initial population of the colony.

When the population was first recorded, there were 100 ground-cuckoos.

c Comparing

$\log_{10}(P) = \log_{10}(a) + t\log_{10}(b)$

to

$\log_{10} P = 0.01t + 2$

gives:

$t\log_{10}(b) = 0.01t$

$b = 10^{0.01}$

$= 1.023...$

$= 1.023$ (3 d.p.)

d Substituting $a = 100$ and $b = 10^{0.01}$ into $P = ab^t$ gives;

$P = 100 \times \left(10^{0.01}\right)^t = 100 \times 10^{0.01t}$

When $t = 30$:

$P = 100 \times 10^{0.01(30)}$

$= 100 \times 10^{0.3}$

$= 199.526...$

$= 200$ (3 s.f.)

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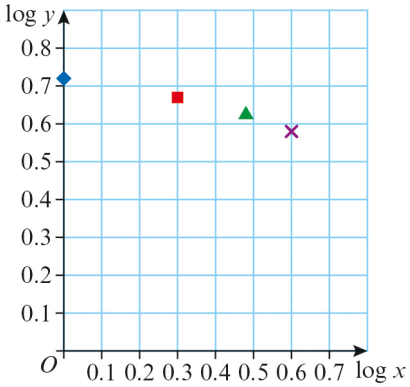
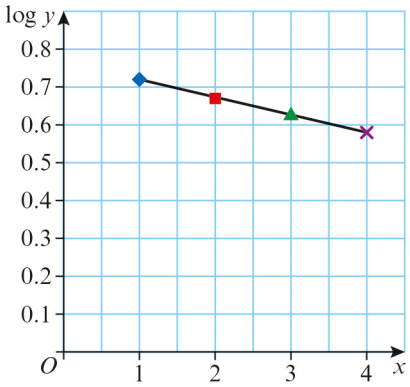
Challenge

$\log x$	0	0.30	0.48	0.60
$\log y$	0.72	0.67	0.63	0.58



The relationship between $\log x$ and $\log y$ is not linear so the relationship is perhaps $y = ax^n$

x	1	2	3	4
$\log y$	0.72	0.67	0.63	0.58



The second graph, $\log y$ against x, is a linear relationship so the relationship is of the form
$y = ab^x$
$\log y = \log (ab^x)$
$\log y = \log a + \log b^x$
$\log y = \log a + x \log b$
Intercept = 0.75
$\log a = 0.75$
$a = 10^{0.75} = 5.8$

$$\text{Gradient} = \frac{0.58 - 0.72}{4 - 1} = -\frac{0.14}{3} = -0.04666...$$

$\log b = -0.04666...$
$b = 10^{-0.04666...}$
$= 0.90$

So the formula is $y = 5.8 \times 0.9^x$

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